Bounds for the Clique Cover Width of Factors of the Apex Graph of the Planar Grid

Farhad Shahrokhi
Department of Computer Science and Engineering, UNT
farhad@cs.unt.edu

Abstract

The *clique cover width* of G, denoted by $ccw(G)$, is the minimum value of the bandwidth of all graphs that are obtained by contracting the cliques in a clique cover of G into a single vertex. For $i = 1, 2, ..., d$, let G_i be a graph with $V(G_i) = V$, and let G be a graph with $V(G) = V$ and $E(G) = \cap_{i=1}^{d}(G_i)$, then we write $G = \cap_{i=1}^{d} G_i$ and call each $G_i, i = 1, 2, ..., d$ a factor of G. The case where G_1 is chordal, and for $i = 2, 3..., d$ each factor G_i has a "small" $ccw(G_i)$, is well studied due to applications. Here we show a negative result. Specifically, let $\hat{G}(k, n)$ be the graph obtained by joining a set of k apex vertices of degree n^2 to all vertices of an $n \times n$ grid, and then adding some possible edges among these k vertices. We prove that if $\hat{G}(k, n) = \cap_{i=1}^{d} G_i$, with G_1 being chordal, then, $max_{2\leq i\leq d}\{ccw(G_i)\} = \Omega(n^{\frac{1}{d-1}})$. Furthermore, for $d = 2$, we construct a chordal graph G_1 and a graph G_2 with $ccw(G_2) \leq \frac{n}{2} + k$ so that $\hat{G}(k, n) = G_1 \cap G_2$. Finally, let $\hat{G}$ be the clique sum graph of $\hat{G}(k_i, n_i)$, where for $i = 1, 2, ...t$, the underlying grid is $n_i \times n_i$ and the sum is taken at apex vertices. Then, we show $\hat{G} = G_1 \cap G_2$, where, G_1 is chordal and $ccw(G_2) \leq \sum_{i=1}^{t}(n_i + k_i)$. The implications and applications of the results are discussed, including addressing a recent question of David Wood.

1 Introduction

In this paper, $G = (V(G), E(G))$ denotes an undirected graph. A *chordal graph* is a graph with no chordless cycles [3]. An *interval graph* is the intersection graph of the intervals on the real line [20]. A *comparability graph* is a graph whose edges have a transitive orientation. An *incomparability graph* [6] is the complement of a comparability graph. A clique cover C in G is a partition of $V(G)$ into cliques. Throughout this paper, we will write $C = \{C_0, C_1, ..., C_t\}$ to indicate that C is an ordered set of cliques in G.

Let $L = \{v_0, v_2, ..., v_{n-1}\}$ be a linear ordering of vertices in G. The *width* of L, denoted by $w(L)$, is $max_{v_i v_j \in E(G)} |j - i|$. The *bandwidth* of G denoted by $bw(G)$, is the smallest width of all linear orderings of $V(G)$ [4]. For a clique cover $C = \{C_0, C_1, ..., C_t\}$, in G, let the *width* of C, denoted by $w(C)$, denote $\max\{|j - i| | xy \in E(G), x \in C_i, y \in C_j, C_i, C_j \in C\}$. The *clique cover width* of G denoted by $ccw(G)$, is the smallest width all ordered clique covers in G[13]. Note that $ccw(G) \leq bw(G)$. It is easy to verify that any graph with $ccw(G) = 1$ is an incomparability graph.

Let $d \geq 1$, be an integer, and for $i = 1, 2, ..., d$ let G_i be a graph with $V(G_i) = V$, and let G be a graph with $V(G) = V$ and $E(G) = \cap_{i=1}^{d}(G_i)$. Then we say G is the *edge intersection graph* of $G_1, G_2, ..., G_d$, and write $G = \cap_{i=1}^{d} G_i$. In this setting, each $G_i, i = 1, 2, ..., d$ is a *factor* of G. Roberts [10] introduced the *boxicity* and *cubicity* of a graph G as the smallest integer d so that G is the edge intersection graph of d interval, or unit interval graphs, respectively. For more recent work see [17, 18, 19].

For integers $d \geq 2$ and $w \geq 1$, let $\mathcal{C}(d, w)$ be the class all graphs G so that $G = \cap_{i=1}^{d} G_i$, where G_1 is chordal, and for $i = 2, 3, ..., d, ccw(G_i) \leq w$. A function that assigns non-negative values to subgraphs of G, is called a measure, if the following hold. (i), $\mu(H_1) \leq \mu(H_2)$, if $H_1 \subseteq H_2 \subseteq G$, (ii) $\mu(H_1 \cup H_2) \leq \mu(H_1) + \mu(H_2)$, if $H_1, H_2 \subseteq G$, (iii) $\mu(H_1 \cup H_2) = \mu(H_1) + \mu(H_2)$, if there are no edges between H_1 and H_2.

Motivated by the earlier results of Chan [5] and Alber and Fiala [1] on geometric separation of fat objects (with respect to a measure), we have derived in [12] a general combinatorial separation Theorem whose statement without a proof was announced in [13].

Theorem 1.1 *([13, 12])*

Let μ be a measure on $G = (V(G), E(G))$, and let $G_1, G_2, ..., G_p$ be graphs with $V(G_1) = V(G_2) =, ..., V(G_p) = V(G)$, $d \geq 2$ and $E(G) = \cap_{i=1}^{p} E(G_i)$ so that G_1 is chordal. Then there is a vertex separator S in G whose removal separates G into two subgraph so that each subgraph has a measure of at most $2\mu(G)/3$. In addition, the induced graph of G on S can be covered with at most $O(2^d l^{*\frac{d-1}{d}} \mu(G)^{\frac{d-1}{d}})$ many cliques from G, where $l^* = max_{2 \leq i \leq d} ccw(G_i)$.

A number of geometric applications of Theorem 1.1 were announced in [13], however, in all these cases G_1 happened to be an interval graph, and hence the full power of the separation theorem was not utilized. We remark that a slightly stronger version of the above Theorem has been proved in [12], where $l^* = \Pi_{i=2}^{d} ccw(G_i)^{\frac{1}{d-1}}$.

Recall that a *tree decomposition* [11, 2] of a graph G is a pair (X, T) where T is a tree, and $X = \{X_i | i \in V(T)\}$ is a family of subsets of $V(G)$, each called a bag, so that the following hold:

- $\cup_{i \in V(T)} X_i = V(G)$
- for any $uv \in E(G)$, there is an $i \in V(T)$ so that $v \in X_i$ and $u \in X_i$.
- for any $i, j, k \in V(T)$, if j is on the path from i to k in T, then $X_i \cap X_k \subseteq X_j$.

Width of (X, T), is the size of largest bag minus 1. Treewidth of G, or $tw(G)$ is the smallest width, overall tree decompositions of G. In an attempt to explore the structure of class $\mathcal{C}(2, w)$ we proved Theorem 1.2 in [15].

Theorem 1.2 *(Universal Representation Theorem:[15]) Let G be a graph and let $L = \{L_1, L_2, ..., L_k\}$ be a partition of vertices, so that for any $xy \in E(G)$, either $x, y \in L_i$ where $1 \leq i \leq k$, or, $x \in L_i, y \in L_{i+1}$, where, $1 \leq i \leq k-1$. Let (X, T) be a tree decomposition of G. Let $t^* = \max_{i=1,2,...,k}\{|L_i \cap X_j||j \in V(T)\}$. (Thus, t^* is the largest number of vertices in any element of L that appears in any bag of T). Then, there is a chordal graph H_1 and a graph H_2 with $ccw(H_2) \leq 2t^* - 1$ so that $G = G_1 \cap G_2$.*

Noting that for any planar graph G, the parameter t^* is at most 4, it follows that any planar G is in class $\mathcal{C}(2, 7)$ [15] (with G_1 being a chordal graph). Consequently, the planar separation theorem [9]follows from Theorem 1.1. We further speculated in [15] that similar results would hold for graphs drawn of surfaces, and graphs excluding specific minors.

Concurrent with our work in [15], in 2013, and independently from us, Dujmović, Morin, and Wood [8], formalized the notation of t^*, and introduced a parameter called the *layered tree width*, or $ltw(G)$, which is the minimum value of t^*, over all tree decompositions and layer partitions of G. They provided significant applications in graph theory and graph drawing, under the requirement that $ltw(G)$ is bounded by a constant. Among other results, it was shown in [8] that $ltw(G) \leq 2g + 3$, for any genus G graph, and as a byproduct some well known separation theorem followed with improved multiplicative constants. Dujmović, Morin, and Wood further classified those graphs G with bounded $ltw(G)$ to be the class of H minor free graphs, for a fixed apex graph H. (H is an apex graph, if $H - \{x\}$ is a planar graph for some vertex x.) Combining this result and the Universal Representation Theorem, it can be concluded that for any fixed apex graph H, there is an integer $w(H)$, so that any $H-$minor free graph G, is in class $\mathcal{C}(2, w(H))$. Using the terminology in the abstract, let $\hat{G}(1, n)$, be the graph obtained from $n \times n$ grid by adding a new vertex of degree n^2 which is adjacent to all vertices of the grid. Then, although this graph not have K_6 as a minor, it does not satisfy the requirement for the usage of the framework in [8], since K_5 is not planar. Specifically, as noted in [8], $ltw(\hat{G}(1, n)) = \Omega(n)$, and hence the layered treewidth method is not applicable. Note that the Universal Representation Theorem also fails

to show that $\hat{G}(1,n)$ is in $\mathcal{C}(2,w)$ for any constant w. David Wood [21] in private communications raised the following question: Might it be true that for every H-minor-free graph G, $G \in \mathcal{C}(2,w(H))$, for some constant $w(H)$ depending on H? Particularly, he inquired if this is true for $\hat{G}(1,n)$. Now, let $\hat{G}(k,n)$ be the graph obtained by joining a set X of k (apex) vertices of degree n^2 to all vertices of an $n \times n$ grid, with possible addition of edges among these vertices in X. In Section two we prove that if $\hat{G}(k,n) = \cap_{i=1}^{d} G_i$, where G_1 is chordal, then $max_{2 \leq i \leq d}\{ccw(G_i)\} = \Omega(n^{\frac{1}{d-1}})$, and hence answer Wood's question in negative, by letting $k=1$ and $d=2$. In the positive direction, for $d=2$, we show $\hat{G}(k,n) = G_1 \cap G_2$, so that G_1 is chordal and $ccw(G_2) \leq \frac{n}{2} + k$, where the upper bound of $\frac{n}{2}+k$ is small enough for the effective application of the separation Theorem 1.1. We extend this result to clique sum graphs. Specifically, let $\hat{G}$ be the clique sum graph of $\hat{G}(k_i,n_i), i = 1,2,...t$, where the underlying grid is $n_i \times n_i$ and the sum is taken at apex vertices. Then, we show $\hat{G} = G_1 \cap G_2$, where, G_1 is chordal and $ccw(G_2) \leq \sum_{i=1}^{t}(n_i + k_i)$.

2 Main Results

Theorem 2.1 *Let $\hat{G}(k,n)$ be the graph obtained by joining a set X of k vertices of degree n^2 to all vertices of an $n \times n$ grid, with possible addition of edges among these vertices. Then, the following hold.*
(i) If $\hat{G}(k,n) = \cap_{i=1}^{d} G_i$, where G_1 is chordal, then, $\max_{2 \leq i \leq d}\{ccw(G_i)\} = \Omega(n^{\frac{1}{d-1}})$.
(ii) There is a chordal graph G_1 and a graph G_2 with $ccw(G_2) \leq \frac{n}{2} + k$, so that $\hat{G}(k,n) = G_1 \cap G_2$.

Proof. For (i), let H be the $n \times n$ planar grid. Robertson and Seymour have shown that any chordalization of H has a clique size n/c, for a constant c. Since G_1 restricted to H is chordal, it follows that, G_1 induced to H, must have a clique S of size $n/2c$. It follows that the subgraph of $\hat{G}(k,n)$ induced on S has a independent set S' of size at least $n/2c$. Now let $G' = \cap_{i=2}^{d} G_i$, and observe that S' must also be an independent set in G', since $\hat{G}(k) = G_1 \cap G'$, and S' is a clique in G_1. Let $\hat{S} = S \cup \{x\}$ for some $x \in X$. Next for $i = 2,3,...,d$, assume that C_i is a clique cover in G_i with $w(C_i) = ccw(G_i)$ and let B_i be the restriction of C_i to $\hat{S}$, and let $2 \leq j \leq d$ so that $|B_j| = max_{2 \leq i \leq d}\{|B_i|\}$. Note that $|(B_j)| \leq 2w(B_j)$, since x is adjacent to all vertices in S'. Now if $|(B_j)| \geq \frac{n^{\frac{1}{d-1}}}{(2c)^{\frac{1}{d-1}}}$, then the claim follows. So assume that $|(B_j)| < \frac{n^{\frac{1}{d-1}}}{(2c)^{\frac{1}{d-1}}}$, then S' can be covered with strictly less than $|B_j|^{d-1} = \frac{n}{2c}$ cliques in G' which is a contradiction,

since S' is independent in G' and $|S'| \geq \frac{n}{2c}$.

For (ii), take the vertices of the grid H in every two consecutive rows, and make them into one single clique, by the addition of edges. This way we get a graph H' which is a unit interval graph. Now make the set X a clique, and add this clique and all $k.n^2$ edges between X and vertices of H to edges of H' to obtain a graph G_1 which can be shown to be chordal. To construct G_2, take any column of H, and make all vertices in this column into one single clique. This way, we obtain a graph I with a clique cover, $O = \{C_1, C_2, ..., C_n\}$, where the vertices in each clique in O are the vertices in a column of H. It is easily seen that $ccw(I) = 1$. To obtain G_2, add to I all vertices in X and edges incident to them. To complete the proof place each vertex $x \in X$ as a clique between $C_{\frac{n}{2}}$, and $C_{\frac{n}{2}+1}$. Observe that $\hat{G}(k, n) = G_1 \cap G_2$, and $ccw(G_2) \leq \frac{n}{2} + k$. $\Box$

Remark 2.1 *Let $G, |V(G) = N|$ be an H minor free graph, where H is a fixed graph. It is known that $tw(G) = O(\sqrt{N})$ [7], consequently by the Universal Representation Theorem, $G = G_1 \cap G_2$, where G_1 is chordal and $ccw(G_2) = O(\sqrt{N})$. Moreover, the upper bound of $O(\sqrt{N})$ is tight, since $\hat{G}(1, n)$ has $N = n^2+1$ vertices, is k_6 minor free and, by Part (i) in Theorem 2.1, if $\hat{G}(1, n) = G'_1 \cap G'_2$, where G'_1 is chordal, then $ccw(G'_2) = \Omega(n)$.*

Remark 2.2 *By Part (i), $\hat{G}(k, n) \notin \mathcal{C}(d, w)$, for any constants w and d. Nonetheless, the upper bound for $ccw(G_2)$ in Part (ii) of Theorem 2.1 is sufficient to use Theorem 1.1 and show that $\hat{G}(k, n), |V(\hat{G}(k))| = N$ has a vertex separator of size $O(N^{1-\epsilon})$, with the splitting ratio $1/3 - 2/3$, where N, is the number of vertices in $\hat{G}$, as long as $k = O(n)$. The bound on the separator size is sufficient for many algorithmic purposes, although it may not be the best possible.*

Let G_1 and G_2 be graphs so that $V(G_1) \cap V(G_2)$ is a clique in both G_1 and G_2. Then, the *clique sum* of G_1 and G_2, denoted by $G_1 \oplus G_2$ is a graph G with $V(G) = V(G_1) \cup V(G_2)$, and $E(G) = E(G_1) \cup E(G_2) - E'$, where E' is a subset of edges (possibly empty) in the clique induced by $V(G_1) \cap V(G_2)$. The clique sum of more than two graphs is defined iteratively, using the definition for two graphs. Let $G = G_1 \oplus ....G_2 \oplus ... \oplus G_k$ then we write $G = \oplus_{i=1}^{k} G_i$. Clique sums are intimately related to the concept of the tree width and tree decomposition; Specifically, it is known that if the tree widths of G_1 and G_2 are at most k, then, so is the tree width of $G_1 \oplus G_2$.

Theorem 2.2 *For $i = 1, 2, ..., t$, let $\hat{G}(k_i, n_i)$ denote the apex graph of an $n_i \times n_i$ grid, with an apex set $X_i, |X_i| = k_i$, so that every vertex in X_i is adjacent to all vertices of the $n_i \times n_i$ grid. Let $\hat{G} = \oplus_{i=1}^{t} \hat{G}_{(k_i, n_i)}$, where the clique sum is only taken at apex sets. Then, $\hat{G} = G_1 \cap G_2$, where, G_1 is chordal and $ccw(G_2) \leq \sum_{i=1}^{t}(n_i + k_i)$.*

Proof. We use the construction in part (ii) of Theorem 2.1. Thus, for $i = 1, 2, ..., t$, there is a chordal graph G_1^i and a graph G_2^i with $ccw(G_2^i) \leq \frac{n_i}{2} + k_i$, so that $\hat{G}(k_i, n_i) = G_1^i \cap G_2^i$. Let $G_1 = \oplus_{i=1}^t G_1^i$, and $G_2 = \oplus_{i=1}^t G_2^i$. Then, $\hat{G} = G_1 \cap G_2$. It is easy to verify that G_1 is chordal. To verify the claim concerning $ccw(G_2)$, we use the construction for general graphs, in [16], in a simpler setting. Particularly, let O_i be the clique cover for $G_2^i, i = 1, 2..., t$, where each vertex $x_i \in X_i$ is represented as a clique and has already been placed in the "middle" of O_i. When taking the sum of $\hat{G}_{(k_i, n_i)}$ and $\hat{G}_{(k_{i+1}, n_{i+1})}, i = 1, 2, .., t-1$, we identify the vertices of the clique in X_{i+1} with the clique in X_i, first, and then place the cliques in O_{i+1} (in the same order that they appear in O_{i+1}) so that each clique in O_i appears immediately to the left of a clique in O_{i+1}. $\square$

We finish this section by exhibiting graphs in $\mathcal{C}(2, 1)$ with arbitrary large layered tree width. For these graphs, Theorem 1.1 is applicable, directly, but the Universal Representation Theorem would fail to be effective.

Example 2.1 (i) *For positive integers n, k, there is an incomparability graph $G \in \mathcal{C}(2, 1)$ of diameter $\Theta(k)$ on $\Theta(nk)$ vertices with $ltw(G) = \Omega(n/k)$.*
(ii) *For positive integers n, k, there is a graph $G \in \mathcal{C}(2, 1)$ with n^2k vertices of diameter $\Theta(n)$ so that $ltw(G) = \Omega(k/n)$.*

For (i), let G be a graph with $ccw(G) = 1$, or a unit incomparability graph on $N = n.k$ vertices, of diameter $k + 2$, where each maximal clique has n vertices. So there is a clique cover $\{C_1, C_2, ..., C_k\}$ of G so that any edge is either in the same clique or joins two consecutive cliques. Let $G_2 = G$. Now let G_1 be a graph which is obtained by adding all possible edges between two consecutive cliques of G, then G_1 is an interval graph. Note that, $G = G_1 \cap G_2$, and $n \leq tw(G) \leq ltw(G)k$.

For (ii) , let H be $n \times n$ grid embedded in the plane. To obtain G replace any vertex $x \in V(G)$ by a clique c_x of k vertices. Now for any $x, y \in V(H)$ with $xy \in E(H)$ place k^2 edges joining every vertex in c_x to very vertex in c_y in G. To obtain G_1, take every two consecutive rows in H, and make all vertices of G in them into one single clique in G_1. It is easy to verify that G_1 is an interval graph. To construct G_2, take any column of H, and make all vertices of G in them into one single clique in G_2. It can be verified that $ccw(G_2) = 1$, and that $G = G_1 \cap G_2$. $\square$

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